Braids and the Yang–Baxter equation

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Contents

1 Interacting particles

To motivate the Yang–Baxter equation, we will start with the simplest version which is called the set-theoretic Yang–Baxter equation. Once we understand this equation, we'll move onto more general version.

1.1 Two particles

Imagine two particles on a line, each with a state that can be any element of a set $\mathcal{X}$. Suppose that the only way particles can change their states is by interacting with each other. An interaction occurs when two particles pass by each other. We could define a function $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ that describes how the states change after interaction. Specifically, if the first particle is in state x and the second particle is in state y, then their states after interacting will be

$$F(x, y) = (F_a(x, y), F_b(x, y)) = (\text{new state of particle 1}, \text{new state of particle 2}),$$

where $F_a, F_b : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ are the components of F. Recall that the particles move past each other when they interact. Thus, to keep track of the whole system we need an element of $\mathcal{X} \times \mathcal{X}$ to keep track of the states and a permutation $\sigma \in S_2$ to keep track of the positions.

1.2 Three particles

Now suppose that we have 3 particles labelled $1, 2, 3$. As before, each particle has a state in $\mathcal{X}$. We can thus keep track of the state of each particle with an element of $\mathcal{X}^3$. The particles also have a position which is described by a permutation $\sigma \in S_3$. The order the entries of $(x, y, z) \in \mathcal{X}^3$ corresponds to the labels of the particles *not their positions*. A possible configuration is shown in Figure 1.

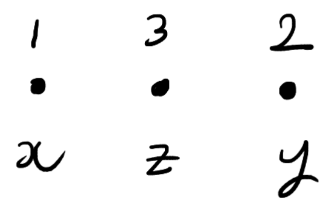

Figure 1: A possible configuration of the three particles. The above configuration can be described by the states $(x, y, z) \in \mathcal{X}^3$ and the permutation $132 \in S_3$.

As before, the particles can interact with each other. However, we'll now add the restriction that the particles can only interact two at a time and interacting particles must have adjacent positions. When two particles interact, they swap positions and their states change according to F. The state and position of the remaining particle is unchanged. For example, in Figure 1 we could interact particles 1 and 3. This will produce the configuration show in Figure 2.

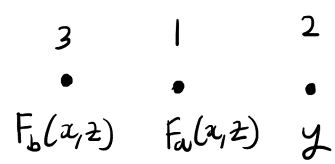

Figure 2: The new configuration after interacting particles 1 and 3 in Figure 1. The configuration is now described by the states $F_{13}(x, y, z) \in \mathcal{X}^3$ and the permutation $312 \in S_3$.

To keep track of how the states of the particles change over time we will introduce three functions from $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ to $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$. These functions are F_{12}, F_{13}, F_{23}. The function F_{ij} is given by applying F to the i, j coordinates of (x, y, z) and acting by the identity on the remaining coordinate. In symbols,

$$
\begin{aligned}
F_{12}(x, y, z) &= (F(x, y), z), \\
F_{13}(x, y, z) &= (F_a(x, z), y, F_b(x, z)), \\
F_{23}(x, y, z) &= (x, F(y, z)).
\end{aligned}
$$

The function F_{ij} exactly describes how the states of the three particles change when particles i and j interact. Now suppose that three particles begin in position 123 and states (x, y, z). We cannot directly interact particles 1 and 3 since they are not adjacent. There are two ways we can pass through particle 2 to make these particles interact:

$$\overset{123}{(x,y,z)} \mapsto F_{23}\overset{132}{(x,y,z)} \mapsto F_{13}(F_{23}\overset{312}{(x,y,z)}) \mapsto F_{12}(F_{13}(F_{23}\overset{321}{(x,y,z)})),$$

and

$$\overset{123}{(x,y,z)} \mapsto F_{12}\overset{213}{(x,y,z)} \mapsto F_{13}(F_{12}\overset{312}{(x,y,z)}) \mapsto F_{23}(F_{13}(F_{12}\overset{321}{(x,y,z)})) \ .$$

Note that after both of these chains of interactions the particles are in the same final position. The function F is said to solve the Yang–Baxter equation if the resulting states are also the same.

Definition 1. A function $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is *a solution to the set theoretic Yang–Baxter equation* if,

$$F_{12} \circ F_{13} \circ F_{23} = F_{23} \circ F_{13} \circ F_{12}. \tag{1}$$

This equation can be visualized as the "braid relation" shown in Figure 3. Here the strings represent the three particles and interacting two particles corresponds to crossing one string over the other.

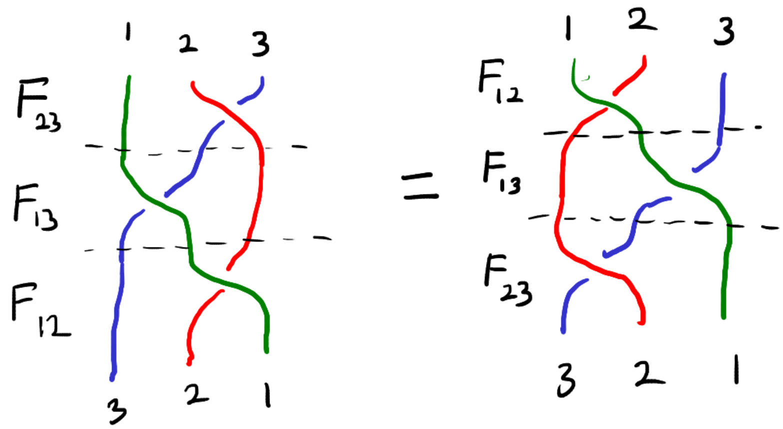

Figure 3: The Yang–Baxter equation can be expressed as the above braid relation.

Here are some solutions to the Yang–Baxter equation.

Example 1. The identity id : $\mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$. This means the particles states don't change after an interaction.

Example 2. The swap map $P : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ given by $P(x, y) = (y, x)$. The states swap after an interaction.

Example 3. Let ρ be a permutation of $\mathcal{X}$, then the map $F(x, y) = (\rho(x), \rho^{-1}(y))$ is a solution to (1) and both sides are equal to the function,

$$(x, y, z) \mapsto (\rho^2(x), y, \rho^{-2}(z)).$$

1.3 A comment

During the talk, it was asked if F had to be symmetric. That is, do we require $F_a(x, y) = F_b(y, x)$. The answer is no. To see why, it is helpful to consider the definition of F_{21}. The definition of F_{21} is act by F on particle 2 and particle 1 and by the identity on particle 3. Thus,

$$F_{21}(x, y, z) = (F_b(y, x), F_a(y, x), z).$$

This is illustrated in Figure 4. Also shown in Figure 4 is applying F_{12} when the particles are in position 123 and have states (y, x, z). We see that the expected symmetry holds for any F.

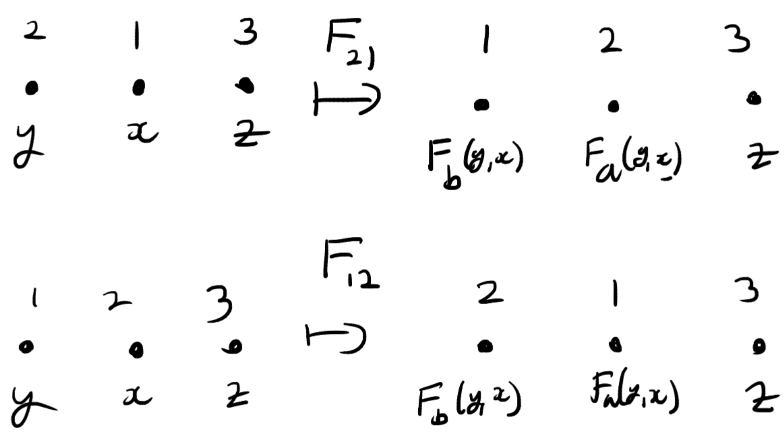

Figure 4: An explanation of the definition F_{21}. Note the symmetry between $F_{21}(x, y, z)$ and $F_{12}(y, x, z)$.

1.4 More particles and the braid group

In the previous section we saw a connection between the Yang–Baxter equation and braids. We'll next more carefully define the n strand braid group and discuss how solutions to the Yang–Baxter equation can be used to describe the interactions between n particles.

The n-stand braid, B_n, consists of configurations of n strings oriented from top to bottom. Two configurations are considered equal if they are topologically the same. Given two braids b_1, b_2, we can glue the top of b_1 to the bottom of b_2 and get a new braid $b_1 b_2$. Two braids in B_4 and their product are shown in Figure 5.

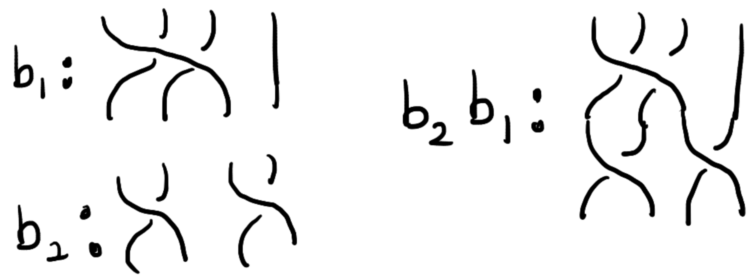

Figure 5: Two braids and their product in the 4 strand braid group.

This operation turns B_n into a group. Note there is a natural group homomorphism from B_n to S_n given by looking at final order of the strings.

The group B_n has generators $s_1, s_2, \ldots, s_{n-1}$ where s_i passes string i over string $i+1$ and leaves the others unchanged. These generators satisfy the following relations,

$$s_i s_j = s_j s_i \text{ for } |i-j| > 1, \tag{2}$$

and

$$s_{i+1} s_i s_{i+1} = s_i s_{i+1} s_i, \text{ for } i = 1, \ldots, n-2. \tag{3}$$

This second relation is exactly the braid relation the Yang–Baxter equation solves. It turns out that the generators $s_1, \ldots, s_{n-1}$ and the above relations give a presentation for B_n. That is, any topological equality of braids can be derived from writing out each braid as a product of s_i's and s_i^{-1}'s and then applying some sequence of the above relations. Furthermore, there exist algorithms that determine if two such words represent the same braid. Some such algorithms have run times that are quadratic in the length of the words.

Let's now suppose that we have n particles with states in $\mathcal{X}^n$ and adjacent interactions given by F. A given sequence of adjacent interaction will correspond to a braid in B_n. If F satisfies the Yang–Baxter equation (1), then equivalent braids will have the same final states.

2 Entangled states

As mentioned at the start of this document, the set theoretic Yang–Baxter equation is the simplest version of the Yang–Baxter equation. It is simple in the sense that the states $\mathcal{X}^3$ were "factorizable". We could project from $\mathcal{X}^3$ to any of the coordinates to recover the state of each particle. A "quantum" version of the Yang–Baxter equation would allow for states that are non-factorizable. Such states are called "entangled."

To do this we will replace the set $\mathcal{X}$ with a complex vector space V. Next, we'll replace the Cartesian product $\mathcal{X} \times \mathcal{X}$ with the tensor product $V \otimes V$. As before we think of $V \otimes V$ as describing the states of two particles. However, we are now allowing for states to be entangled. The general form of an element in $V \otimes V$ is

$$\sum_i x_i \otimes y_i,$$

which may not factor as $x \otimes y$. If such a sum does not factor, there is no natural way to project it onto two components.

The function $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ that described how states changed is replaced with a linear map $R : V \otimes V \to V \otimes V$. As before, we can define three maps $R_{12}, R_{13}, R_{23} : V \otimes V \otimes V \to V \otimes V \otimes V$ by acting by R on two out of three of the terms. For example, if $R(x \otimes z) = \sum_i x_i \otimes z_i$, then

$$R_{13}(x \otimes y \otimes z) = \sum_i x_i \otimes y \otimes z_i,$$

and R_{13} is extended linearly to the rest of $V^{\otimes 3}$. As before, we want R to respect the braid relation when we interact particles. This leads to the more general Yang–Baxter equation.

Definition 2. A linear map $R : V \otimes V \to V \otimes V$ is *a solution to the Yang–Baxter equation* if,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{4}$$

Such a map is called an *r-matrix*.

Remark 1. Suppose V has dimension N. Then R can be written as a $N^2 \times N^2$ matrix and (4) is an equality of two $N^3 \times N^3$ matrices. Thus, the Yang–Baxter equation is a set of N^6 homogeneous cubic equations in N^4 variables.

Example 4. If $\mathcal{X}$ is a basis for V, then any function $F : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ extends to a linear map $R : V \otimes V \to V \otimes V$. The resulting linear map solves the Yang–Baxter equation if and only if F solve the set theoretic Yang–Baxter equation. Thus, Examples 1-3 have corresponding r-matrices.

Example 5. Suppose V is two-dimensional with basis $\{a, b\}$. Define $R : V \otimes V \to V \otimes V$ on the basis $\{a \otimes a, a \otimes b, b \otimes a, b \otimes b\}$ by,

$$\begin{aligned}
R(a \otimes a) &= a \otimes a \\
R(a \otimes b) &= q(a \otimes b) \\
R(b \otimes a) &= (1 - q^2)(a \otimes b) + q(b \otimes a) \\
R(b \otimes b) &= b \otimes b.
\end{aligned}$$

The parameter q can be any complex number and the above map will always solve the Yang–Baxter equation. This example is from [1], but they use the form of the braid group given in equation (7) at the end of this document.

3 The parametric Yang–Baxter equation

Another variation on the Yang–Baxter equation is the *parametric Yang–Baxter equation*. In this setting, we allow for each particle to have a parameter $u \in \Theta$. Unlike states, parameters do not change when particles interact with each other. However, the parameters affect the particle interactions. This means that for every pair of parameters $(u_1, u_2) \in \Theta$, we have a linear map $R(u_1, u_2) : V \otimes V \to V \otimes V$ that determines how the states of two particles change after interaction when the particles have parameters u_1 and u_2. The parametric Yang–Baxter equation states that the braid relation holds for all combinations of parameters.

Definition 3. A collection of linear maps $R(u_1, u_2) : V \otimes V \to V \otimes V$ is *a solution to the parametric Yang–Baxter equation* if, for all $(u_1, u_2, u_3) \in \Theta^3$,

$$R_{12}(u_1, u_2)R_{13}(u_1, u_3)R_{23}(u_2, u_3) = R_{23}(u_2, u_3)R_{13}(u_1, u_3)R_{12}(u_1, u_2). \tag{5}$$

Example 6. Often $\Theta = \mathbb{R}$ and $R(u_1, u_2) = R(u_1 - u_2)$ for map R. In this case (5) becomes,

$$R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u),$$

for all $u, v \in \mathbb{R}$. If V is any vector space and $P : V \otimes V \to V \otimes V$ is given by $P(x \otimes y) = y \otimes x$, then $R(u) = P + uI$ is a solution to the parametric Yang–Baxter equation (5). According to [2], this was an example Yang was interested in.

Example 7. Suppose that V has a basis $\mathcal{X}$ such that $\{x \otimes y : x, y \in \mathcal{X}\}$ is invariant under $R(u_1, u_2)$ for every choice of parameters u_1, u_2. Then each linear function $R(u_1, u_2)$ can be described by a function $F^{u_1,u_2} : \mathcal{X} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$. The collection of linear function $R(u_1, u_2)$ solve (5) if and only if

$$F_{12}^{u_1,u_2} \circ F_{13}^{u_1,u_3} \circ F_{23}^{u_2,u_3} = F_{23}^{u_2,u_3} \circ F_{13}^{u_1,u_3} \circ F_{12}^{u_1,u_2}, \tag{6}$$

for all $(u_1, u_2, u_3) \in \Theta^3$. Equation (6) is called the *parametric set theoretic Yang–Baxter equation*. Here is a solution from [3] with $\mathcal{X} = \Theta = \mathbb{R}_{++}$,

$$F^{\alpha,\beta}(x, y) = \left(y\frac{1 + \beta xy}{1 + \alpha xy}, x\frac{1 + \alpha xy}{1 + \beta xy} \right).$$

Remark 2. The parametric form of the Yang–Baxter equation (5) is equivalent to the non-parametric version on an enlarged state space. Suppose our state space is V and the parameter space is Θ. Instead of considering the states and parameters separately we can combine them into one space. The natural way to do this is to define $\widetilde{V} = V \otimes (\mathbb{C}^{\Theta})$ where $\mathbb{C}^{\Theta}$ is the direct sum over Θ of $\mathbb{C}$ – meaning Θ is a basis for $\mathbb{C}^{\Theta}$.

Recall the principle that when two particles interact, their parameters determine how the states change but the parameters themselves do not change. We can translate this principle into a linear map from $\widetilde{R} : \widetilde{V} \otimes \widetilde{V} \to \widetilde{V} \otimes \widetilde{V}$ by,

$$\widetilde{R}(\underbrace{x \otimes y}_{\text{states}} \otimes \underbrace{u_1 \otimes u_2}_{\text{parameters}}) = \underbrace{R(u_1, u_2)(x \otimes y)}_{\text{new states}} \otimes \underbrace{u_1 \otimes u_2}_{\text{parameters}}.$$

To define $\widetilde{R}$ we have implicitly used the isomorphism $\widetilde{V} \otimes \widetilde{V} \cong V^{\otimes 2} \otimes (\mathbb{C}^{\Theta})^{\otimes 2}$. The same principle shows that solutions to the parametric set theoretic Yang–Baxter equation give solutions to the non-parameter version on the set $\widetilde{X} = \mathcal{X} \times \Theta$.

4 Braided form of the Yang–Baxter equation

When we multiply braids b_1 and b_2, the braid b_1 acts on the braid b_2 based on the final position of the strings. This is in contrast with the linear functions R_{ij}. In the composition $R_{13}R_{23}$, the second function represents the interaction between the particles labelled 1 and 3, not the particles in positions 1 and 3.

Define a new function $\check{R} = PR$ where, as before, $P(x \otimes y) = y \otimes x$. Suppose we have a particle in state x on the left of a particle in state y. If these two particles interact, they will swap positions and the state will change to $R(x \otimes y)$. The tensor $\check{R}(x \otimes y)$ also describes the new state but now the first component corresponds to the particle in position 1 not the particle labelled 1.

If we think of $v \in V^{\otimes n}$ as representing the states of the particles *ordered by their positions*. Then $\check{R}_{ii+1}(v)$ will give the states of the particles ordered by their positions after particles in positions i and $i+1$ have interacted. One can check that R satisfies the Yang–Baxter equation (4) if and only if $\check{R}$ satisfies,

$$\check{R}_{23}\check{R}_{12}\check{R}_{23} = \check{R}_{12}\check{R}_{23}\check{R}_{12}.$$

Which is more often written as,

$$(I \otimes \check{R})(\check{R} \otimes I)(I \otimes \check{R}) = (\check{R} \otimes I)(I \otimes \check{R})(\check{R} \otimes I). \tag{7}$$

Equation (7) is also called the Yang–Baxter equation and solutions to the two equations are in bijection with each other. If $\check{R}$ is invertible and solves (7), then the map $s_i \to \check{R}_{ii+1}$ defines a representation of B_n on $V^{\otimes n}$.

References

[1] John Baez. *The Yang-Baxter Equation*. URL: https://math.ucr.edu/home/baez/braids/node4.html.

[2] Michio Jimbo. *Introduction to the Yang-Baxter equation*. 1989. URL: https://www.worldscientific.com/doi/abs/10.1142/S0217751X89001503.

[3] Makiko Sasada and Ryosuke Uozumi. *Yang-Baxter maps and independence preserving property*. 2022. DOI: 10.48550/ARXIV.2212.00963. URL: https://arxiv.org/abs/2212.00963.